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An Exact 2-Point Distribution for Burger's Turbulence *

P. Gräff, H. Tasso, and J. Steuerwald

Max-Planck-Institut für Plasmaphysik, 8046 Garching bei München, Federal Republic of Germany

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An exact possible turbulence for Burger's equation is found. It allows the 2-point distribution function to be calculated in closed form.

Burger's equation offers a solvable model of hydrodynamics. This would be of great value to strong turbulence as a means of testing approximation procedures or numerical methods if the statistical initial value problem could also be treated. However, rigorous results are difficult to obtain since one is dealing with a Liouville equation for the infinitely many degrees of freedom of a continuum. (A rigorous description of this kind of problem was first given by Hopf [1] 1952.) The special case for the spatial white noise at $t=0$ was sketched earlier [2]. An admissible 1-point distribution for all times has already been found [3]. It is of Lorentz form:

$$P_1(v,t) \approx 1/[a(t)+b(t)\,v^2]\,.$$

In this paper we derive the 2-point distribution function in closed form from the Hopf characteristic functional.

Burger's equation is of the form:

$$\partial v/\partial t + v\,\partial v/\partial x - \nu\,\partial^2 v/\partial x^2 = 0\,. \qquad (1)$$

The ansatz $v = -2\nu\;\partial \ln|M|/\partial x$ due to Cole reduces Eq. (1) to the heat conduction equation:

$$\partial M/\partial t = \nu\,\partial^2 M/\partial x^2 \qquad (2)$$

whose relevant solutions are

$$M(x,t) = (1/\sqrt{2\pi t \nu}) \int_{-\infty}^{+\infty} \cdot \exp\{-(x-\xi)^2/2t\nu\}\, M_0(\xi)\, d\xi \qquad (3)$$

and

$$p(x,t) = \langle M_1 M_2\rangle = (1/\sqrt{8\pi t\nu}) \cdot \int_{-\infty}^{+\infty} \exp\{-(x-\xi)^2/8t\nu\}\, p(\xi)\, d\xi\,. \qquad (3^*)$$

Consider the characteristic functional [1] for a Gaussian measure which is conserved by the heat equation. For the simultaneous measure of M and M' we get

$$\varphi_{M,M'}(\alpha,\beta) = \mathcal{E}[\exp\{i\int_{-\infty}^{+\infty} \alpha(x)\,M(x)\,dx + i\int_{-\infty}^{+\infty} \beta(x)\,M'\,dx\}]$$

$$= \mathcal{E}[\exp\{i\int_{-\infty}^{+\infty} (\alpha-\beta')\,M(x)\,dx\}] = \varphi_M(\alpha-\beta') \qquad (4)$$

Reprint requests to Dr. L. Johannsen, Bibliothek, Max-Planck-Institut für Plasmaphysik, *D-8046 Garching* bei München.

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or, more explicitly, for the case of homogeneous turbulence

$$\varphi(\alpha,\beta) = \exp\{-\tfrac{1}{2}\int\int_{-\infty}^{+\infty} p[(x-y),t][\alpha(x)-\beta'(x)][\alpha(y)-\beta'(y)]\,\mathrm{d}x\,\mathrm{d}y\}$$

$$= \exp\{-\tfrac{1}{2}\int\int_{-\infty}^{+\infty} [p(x-y,t)\,\alpha(x)\,\alpha(y) - 2\,p'\,\alpha(x)\,\beta(y) - p''\,\beta(x)\,\beta(y)]\,\mathrm{d}x\,\mathrm{d}y\} \tag{5}$$

where $p' = \partial p/\partial x$.

The joint probability can be obtained from the characteristic functional by Fourier transforming:

$$P'(x_1{:}M_1, M_1'; x_2{:}M_2, M_2', t) = \left(\frac{1}{2\pi}\right)^4 \int_{-\infty}^{+\infty} \exp\{-i[M_1\lambda_1 + M_2\lambda_2 + M_1'\mu_1 + M_2'\mu_2]\} \tag{6}$$

$$\cdot \exp\{-\tfrac{1}{2}\iint_{-\infty}^{+\infty} [P\,\alpha_0(x)\,\alpha_0(y) - 2\,p'\,\alpha_0(x)\,\beta_0(y) - p''\,\beta_0(x)\,\beta_0(y)]\,\mathrm{d}x\,\mathrm{d}y\}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2$$

where

$$\alpha_0(x) = \lambda_1\,\delta(x-x_1) + \lambda_2\,\delta(x-x_2)$$
$$\beta_0(x) = \mu_1\,\delta(x-x_1) + \mu_2\,\delta(x-x_2).$$

Equation (6) becomes:

$$P_2' = \frac{1}{4\pi^2}\,\frac{1}{\sqrt{|R_{ik}|}}\,\exp\{-\tfrac{1}{2}\sum_{i,k=1}^{4} R_{ik}^{-1}\,m_i\,m_k\} \tag{7}$$

where
$$R_{ik} = \begin{pmatrix} p_0 & p & 0 & -p' \\ p & p_0 & p' & 0 \\ 0 & p' & -p_0'' & -p'' \\ -p' & 0 & -p'' & -p_0'' \end{pmatrix} \quad \text{and} \quad m_i = \begin{pmatrix} M_1 \\ M_2 \\ M_1' \\ M_2' \end{pmatrix}$$

and using an algebraic computer code[4] is was pos-sible to calculate easily R_{ik}^{-1} and $|R_{ik}|$

$$R_{ik}^{-1} = \frac{1}{|R_{ik}|}\begin{pmatrix} p_0''^2\,p_0 + p'^2\,p_0'' - p_0\,p''^2, & -[p\,p_0''^2 - p''^2\,p + p'^2\,p''], & -p'(p_0''\,p - p_0\,p''), & [p'(p\,p'' - p_0\,p_0'') - p'^3] \\ & p_0(p_0''^2 - p''^2) + p_0''\,p'^2, & [p'(p_0\,p_0'' - p\,p'') + p'^3], & -p'(p_0\,p_0'' - p\,p_0'') \\ & & p_0''(p^2 - p_0^2) - p_0\,p'^2, & -[p''(p^2 - p_0^2) - p\,p'^2] \\ \text{symmetric} & & & p_0''(p^2 - p_0^2) - p_0\,p'^2 \end{pmatrix}, \tag{8}$$

$$\Delta = |R_{ik}| = (p_0''^2 - p''^2)(p_0^2 - p^2) + 2\,p'^2(p_0\,p_0'' - p\,p'') + p'^4. \tag{8*}$$

Now we can calculate $P_2(x_1{:}v_1, x_2{:}v_2, t)$,

$$P_2 = \int_{-\infty}^{+\infty} P_2'\,\delta(v_1 + 2\,\nu\,M_1'/M_1)\,\delta(v_2 + 2\,\nu\,M_2'/M_2)\,\mathrm{d}M_1\,\mathrm{d}M_2\,\mathrm{d}M_1'\,\mathrm{d}M_2'. \tag{9}$$

Using Eq. (7) for P_2', we obtain:

$$P_2 = \frac{1}{4\pi^2}\,\frac{1}{\sqrt{|R_{ik}|}}\,\frac{1}{4\nu^2}\int_{-\infty}^{+\infty} \exp\left\{-\tfrac{1}{2}\sum_{\alpha,\beta=1}^{2} R'_{\alpha\beta}\,M_\alpha\,M_\beta\right\}|M_1|\,|M_2|\,\mathrm{d}M_1\,\mathrm{d}M_2, \tag{10}$$

$$R'_{\alpha\beta} = \frac{1}{|R_{ik}|}\begin{pmatrix} \left\{\begin{aligned} & p_0(p_0''^2 - p''^2) + p_0''\,p'^2 \\ & + \frac{v_1^2}{4\nu^2}[p_0''(p^2 - p_0^2) - p_0\,p'^2] \\ & + \frac{v_1}{\nu}\,p'(p_0''\,p - p_0\,p'') \end{aligned}\right\} & \left\{\begin{aligned} & 2[p(p''^2 - p_0''^2) - p'^2\,p''] \\ & - [p'(p\,p'' - p_0\,p_0'') - p'^3]\left(\frac{v_2}{\nu} - \frac{v_1}{\nu}\right) \\ & - \frac{1}{2}\,\frac{v_1\,v_2}{\nu^2}[p''(p^2 - p_0^2) - p\,p'^2] \end{aligned}\right\} \\ \text{symmetric} & \left\{\begin{aligned} & p_0(p_0''^2 - p''^2) + p_0''\,p'^2 \\ & + \left(\frac{v_2}{2\nu}\right)^2[p_0''(p^2 - p_0^2) - p_0\,p'^2] \\ & - \frac{v_2}{\nu}\,p'(p_0''\,p - p_0\,p'') \end{aligned}\right\} \end{pmatrix}. \tag{11}$$

The remaining problem is the integration in (10), which is of the type:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{-[\alpha x^2 + 2\beta x y + \gamma y^2]\} \, |x| \, |y| \, dx \, dy \tag{12}$$

with $\alpha = R_1''/2$, $\beta = R_{12}'/2$, $\gamma = R_{22}'/2$.

The integral (12) is also:

$$I = 2\int_0^\infty\int_0^\infty \exp\{-[\alpha x^2 + 2\beta x y + \gamma y^2]\} \, x y \, dx \, dy + 2\int_0^\infty\int_0^\infty \exp\{-[\alpha x^2 - 2\beta x y + \gamma y^2]\} \, x y \, dx \, dy \tag{13}$$

$$= 2 I_1 \qquad\qquad + 2 I_2$$

Introducing new variables and $\lambda = \beta/\sqrt{\alpha\gamma}$, we have

$$I_2 = I_1(-\lambda)$$

and

$$I_1(\lambda) = -\frac{1}{\alpha\gamma}\frac{\partial}{\partial\lambda}\left\{\frac{1}{\sqrt{1-\lambda^2}}\int_0^\infty e^{-x^2} dx \int_{\lambda x/\sqrt{1-\lambda^2}}^\infty e^{-z^2} dz\right\}.$$

Let us call

$$K(a) = \int_0^\infty e^{-x^2} dx \int_{a x}^\infty e^{-z^2} dz$$

then

$$dK/da = -\int_0^\infty e^{-x^2(1+a^2)} x \, dx = -1/2(1+a^2)$$

and

$$K = -\tfrac{1}{2}\operatorname{arctg}(\lambda/\sqrt{1-\lambda^2}) + \pi/4 . \tag{14}$$

Using (10), (12), (13) and (14), we obtain:

$$P_2(x_1{:}v_1, x_2{:}v_2, t) = \frac{1}{8\pi^2\nu^2}\frac{1}{\Delta^{1/2}}\frac{1}{\alpha\gamma}\left[1 + \frac{\lambda \operatorname{arctg}(\lambda/\sqrt{1-\lambda^2})}{(1-\lambda^2)^{3/2}}\right] \tag{15}$$

where Δ, α, β, γ are given in (8*), (11) and (12) and

$$\lambda = \beta/\sqrt{\alpha\gamma} .$$

For $x \to \infty$ we obtain

$$P_2 \approx 1/\big(a(t) + b(t)\, v_1^2\big)\big(a(t) + b(t)\, v_2^2\big)$$

which gives no correlations if $\int_{-\infty}^{+\infty} P_2 \, v_1 \, v_2 \, dv_1 \, dv_2$ is calculated in the sense of a principal part.

Notice also that the toal energy:

$$\int_{-\infty}^{+\infty} P_1(v, t) \, v^2 \, dv \quad \text{diverges}$$

because P_1 is a Lorentzian distribution.

These peculiar results are due to the choice of a Gaussian measure of the solutions of the heat equation. The comparison with numerical turbulence calculations is then only possibel for not too large v_1, v_2.

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